Theorem: For all $n \geq 2$, there is a prime number p such that $n < p < 2n$

Recommended levels for proof: 4

Proof:

The idea is to study the binomial coefficient $\binom{2n}{n}$ since these numbers have nice properties and you will see that this can be exploited to deduce a lot of things, in particular the theorem above.

Lemma 1: Define $R(n,p)$ for a prime number p to be the largest number r such that $\binom{2n}{n}$ is divisible by p^r. Then $p^{R(n,p)} \leq 2n$.

Proof of Lemma 1:

We define floor x or $\lfloor x \rfloor$ to mean the largest integer less than or equal to x. We want to determine how many times p appears in the prime factorization of $n!$. Note that we get one appearance of p for each multiple of p less than m, and there are $\left\lfloor \frac{n}{p} \right\rfloor$ of those. For non-multiples of p less than n nothing happens since there is no p in the prime factorization of those. However, we are not done: For all multiples of p^2 we have only counted one of the two or more p's that appear. Therefore we have an extra $\left\lfloor \frac{n}{p^2} \right\rfloor$ copies of p in the prime factorization, but then for multiples of p^3 we need an extra $\left\lfloor \frac{n}{p^3} \right\rfloor$ copies, and so on. The formula is $\sum_{j=1}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor$. Therefore the number of times p appears in the prime factorization of $\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$ is $\sum_{j=1}^{\infty} \left\lfloor \frac{2n}{p^j} \right\rfloor - 2\sum_{j=1}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor = \sum_{j=1}^{\infty} \left(\left\lfloor \frac{2n}{p^j} \right\rfloor - 2\left\lfloor \frac{n}{p^j} \right\rfloor \right)$

Now each term in the sum is either 0 or 1, because for any arbitrary x, $\lfloor 2x \rfloor - 2\lfloor x \rfloor = 2\lfloor x \rfloor - 2\lfloor x \rfloor$ is 0 whenever the non-integer part of x is strictly less than 0.5, and 1 otherwise (as $\lfloor 2x \rfloor$ is $1 + 2\lfloor x \rfloor$ in that case). But when $j > \log_p(2n)$, both $\left\lfloor \frac{2n}{p^j} \right\rfloor$ and $\left\lfloor \frac{n}{p^j} \right\rfloor$ are 0 so the term is 0, and also $\frac{n}{p^j}$ is less than 0.5 so by the logic above the term is 0. Therefore there are at most $\log_p(2n)$ terms that contribute 1. Since this is the number of powers of p in the prime factorization, so these terms contribute something at most 2n, which is exactly what the lemma claimed.

Lemma 2: No prime factors p of $\binom{2n}{n}$ satisfy $\frac{2n}{3} < p \leq n$ (unless p=2)

Proof of lemma 2:

Since p is at least 3 and greater than $\frac{2n}{3}$, $p^2 > 2n$ so the number of copies of p in (2n)! is $\left\lfloor \frac{2n}{p} \right\rfloor$ which is 2 by definition, but the number of copies in $(n!)^2$ is $2\left\lfloor \frac{n}{p} \right\rfloor = 2 * 1 = 2$, so in $\frac{(2n)!}{(n!)^2}$ we get 2-2=0 copies.

Lemma 3: Define n# as the product of all primes less than or equal to n (1#=1 by convention). Then the lemma says that for all $n \geq 1$, $n\# < 4^n$.

Proof of lemma 4:

We note that 2#=2, so for n=1 and n=2 the lemma holds. We will now use strong induction to prove the lemma. If we are in the case that we have proven it for everything up to an odd number and trying to prove it for the next number which is even, then it is trivial as an even number greater than 2 will not increase the size of n# since it is not prime, so if it was less than 4^n it will stay that way. So lets assume we have proven it for all integers 1 to 2k and we want to prove it for 2k+1. Now consider the binomial coefficient $\binom{2k+1}{k} = \frac{(2k+1)!}{k!(k+1)!}$. All primes from k+2 to 2k+1 inclusive only appear once in the numberator, so the product of all such primes divides $\binom{2k+1}{k}$, and is therefore less than or equal to $\binom{2k+1}{k}$. We see the first hint of primes from a number to double that number which is a good sign that we are headed in the right direction. But $\binom{2k+1}{k} = \binom{2k+1}{k+1}$ by symmetry of the choose function (Choosing k things and not choosing k things can be done in the same number of ways). So we now have the chain of inequalities $\frac{(2k+1)\#}{(k+1)\#} \leq \binom{2k+1}{k} = \frac{1}{2}\left(\binom{2k+1}{k} + \binom{2k+1}{k+1}\right)$. Note now that the total number of ways to choose some things out of 2k+1 things is 2^{2k+1} as for each thing we either choose it or not choose it. Therefore $\binom{2k+1}{k} + \binom{2k+1}{k+1}$ is strictly less than 2^{2k+1}as we have left out a lot of possibilities, such as choosing none of the things, or all of them. Therefore $\frac{(2k+1)\#}{(k+1)\#} < 2^{2k}$. We can rearrange this to get $(2k+1)\# < 4^k(k+1)\# < 4^k 4^{k+1} < 4^{2k+1}$ (by the induction hypothesis), so we are done.

Now we will prove the main theorem.

Lets consider $\frac{4^n}{2n}$. We will suppose there are no primes between n and 2n and then use this to put an upper bound on $\frac{4^n}{2n}$, and we will see that this upper bound is rather restrictive. First of all, notice that $\binom{2n}{n}$ is the largest binomial coefficient of the form $\binom{2n}{k}$. This is kind of obvious but I will prove it rigorously because it's not *completely* obvious. The reason for this is suppose k<n (if k>n the symmetry property means the same conclusion happens), then consider $\frac{\binom{2n}{k}}{\binom{2n}{n}}$. We hope to show this is less than 1. $\frac{\binom{2n}{k}}{\binom{2n}{n}} = \frac{\frac{(2n)!}{k!(2n-k)!}}{\frac{(2n)!}{(n!)^2}} = \frac{(n!)^2}{k!(2n-k)!} = \frac{1*1*2*2*3*3*...*n*n}{1*1*2*2*3*3*...*k*k*(k+1)*(k+2)*...*(2n-k)} = \frac{(k+1)*(k+2)*...*(n-1)*n}{(n+1)*(n+2)*...*(2n-k)}$. So we have n-k terms on both the numerator and the denominator, but each one in the numerator is strictly smaller than the ones on the denominator. But we know from earlier discussion that the sum of all $\binom{2n}{k}$ as k goes from 0 to 2n is 4^n. For all positive integers n, $\binom{2n}{n} \geq 2$, so we can combine the terms $\binom{2n}{0}$ and $\binom{2n}{2n}$ into 2 and we will then be summing 2n terms in which $\binom{2n}{n}$ is still the largest. Therefore if $\binom{2n}{n} < \frac{4^n}{2n}$ we have a problem as that is the largest term so we would be summing 2n terms that are strictly less than $\frac{4^n}{2n}$ and end up with something less than 4^n which is a contradiction. Therefore we start our chain of inequalities with $\frac{4^n}{2n} \leq \binom{2n}{n}$. Since there are no primes between n and 2n by assumption, it means that by lemma 2 all primes dividing $\binom{2n}{n}$ are less than or equal to $\frac{2n}{3}$. That's is quite restrictive. We write $\binom{2n}{n}$ in terms of its prime factors using product notation as $\left(\prod_{p\geq\sqrt{2n}} p^{R(p,n)}\right)\left(\prod_{\sqrt{2n}<p\leq\frac{2n}{3}} p^{R(p,n)}\right)$ where R is as defined above. This is less than $\left(\prod_{p\geq\sqrt{2n}} 2n\right)\left(\prod_{\sqrt{2n}<p\leq\frac{2n}{3}} p\right)$ by lemma 1. This is thus less than $(2n)^{\sqrt{2n}-1}4^{\frac{2n}{3}}$ by lemma 3 and the fact that there are clearly not more than $\sqrt{2n}-1$ primes less than $\sqrt{2n}$. Therefore we have shown that if we

have a counterexample to the postulate, then $\frac{4^n}{2n} < (2n)^{\sqrt{2n}-1}4^{\frac{2n}{3}}$. Multiplying through by 2n gives $4^n < (2n)^{\sqrt{2n}}4^{\frac{2n}{3}}$. Taking $\log_2$ of both sidesgives $2n < \frac{4n}{3} + \sqrt{2n}\log_2(2n)$, so $\frac{2n}{3} < \sqrt{2n}\log_2(2n)$, so now $\sqrt{2n} < 3\log_2(2n)$, so $2n < 9\log_2^2(2n)$. The derivative of the right hand side with respect to n is $\frac{d}{dx}\frac{9\ln^2(2x)}{\ln^2(2)} = \frac{d}{dx}\left(\ln^2(x)\left(\frac{9}{\ln^2(2)}\right) + c\right) = 2\left(\frac{9}{\ln^2(2)}\right)\left(\frac{\ln(x)}{x}\right) = \left(\frac{18}{\ln^2(2)}\right)\left(\frac{\ln(x)}{x}\right)$ by standard differentiation and logarithm rules. We can differentiate again with the quotient rule to get the second derivative. This gives $\left(\frac{18}{\ln^2(2)}\right)\frac{1-\ln(x)}{x^2}$. For all x greater than e, this is negative, so we will have something with downwards curvature that has to beat something linear (ie 2n). One can check by computation that the inequality $2n < 9\log_2^2(2n)$ fails for $n = 427$ but not for $n = 426$, therefore since the curvature is only downward, $9\log_2^2(2n)$ will never again cross the 2n line so the inequality will always fail after that. So any counterexample has $n < 427$. But if $2 \leq n \leq 426$ we can verify that the theorem holds by picking one of the primes 3, 5, 7, 13, 23, 43, 83, 163, 317 or 631 depending on what n is. So done.